# Sublinear Algorithms for Estimating Single-Linkage Clustering Costs




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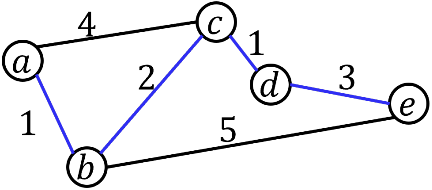

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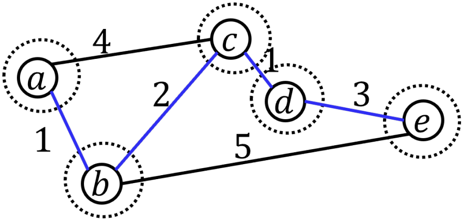

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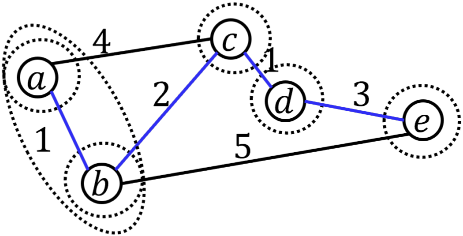

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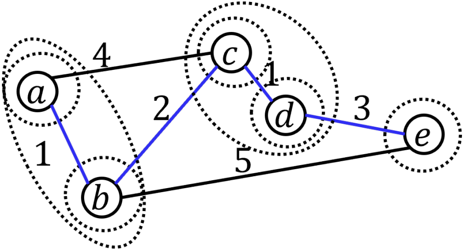

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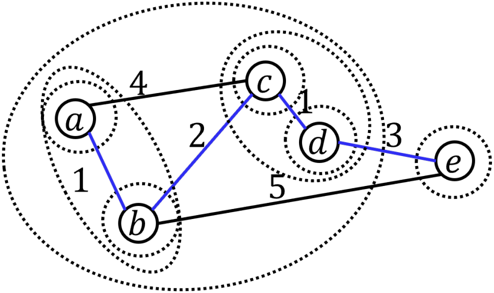

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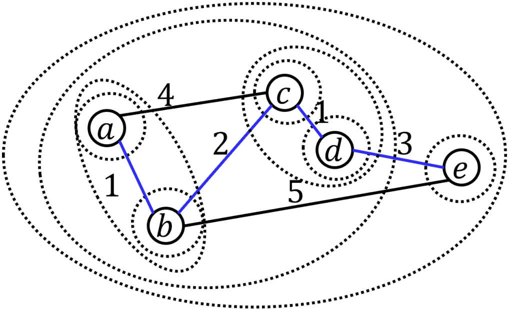

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Naive solution: compute an $\mathrm{MST}$ in $\tilde{O}(nd)$ time
Question: estimate $\mathrm{cost}(G)$ and $\mathrm{cost}_k$ in **sublinear** time?

# Main Results

$W$: max weight  $d$: average degree  query model: adj. list

| Setting | $\mathrm{cost}(G)$ | $\mathrm{cost}_k$ | Lower bound |
|---|---|---|---|
| Distance Case | $\tilde{O}(\frac{\sqrt{W}}{\varepsilon^3} d)$ | $\tilde{O}(\frac{\sqrt{W}}{\varepsilon^3} d)$ | $\Omega(\frac{\sqrt{W}}{\varepsilon^2} d)$ |
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**Succinct** representation of the SLC estimates $(\widehat{\text{cost}}_1, \ldots, \widehat{\text{cost}}_n)$ s.t. $\forall k$, recover $\widehat{\text{cost}}_k$ in a **short** time, and **on average** a $(1+\varepsilon)$ estimate

On average: $\sum_{k=1}^{n} |\widehat{\text{cost}_k} - \text{cost}_k| \leq \varepsilon \cdot \text{cost}(G) = \varepsilon \sum_{k=1}^{n} \text{cost}_k$
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[1] Applying [CRT05], one can get: $(1+\varepsilon)$-estimate, $\tilde{O}(\frac{W}{\varepsilon^2}d)$ queries

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Total running time & queries: $\tilde{O}(\frac{\sqrt{W}}{\varepsilon^3})$